Technical Note: Auto-Regressive Model

We originally composed these technical notes after sitting in on a time series analysis class. Over the years, we've maintained these notes and added new insights, empirical observations and intuitions acquired. We often go back to these notes for resolving development issues and/or to properly address a product support matter.

In this paper, we'll go over another simple, yet fundamental, econometric model: the auto-regressive model. Make sure you have looked over our prior paper on the moving average model, as we build on many of the concepts presented in that paper.

This model serves as a cornerstone for any serious application of ARMA/ARIMA models.

Background

The auto-regressive model of order p (i.e. $AR(p)$) is defined as follows:

$$x_t = \phi_o + \phi_1 x_{t-1} + \phi_2 x_{t-2} + ... + \phi_p x_{t-2} + a_t$$
$$a_t = \varepsilon_t \times \sigma$$
$$\varepsilon_t \sim \text{i.i.d} \sim N(0,1)$$

Where

- a_t is the innovations or shocks for our process
- σ is the conditional standard deviation (aka volatility)

Essentially, the $AR(p)$ is merely a multiple linear regression model where the independent (explanatory) variables are the lagged editions of the output (i.e. $x_{t-1}, x_{t-2}, ..., x_{t-p}$). Keep in mind that $x_{t-1}, x_{t-2}, ..., x_{t-p}$ may be highly correlated with each other.

Why do we need another model?

First, we can think of an AR model as a special (i.e. restricted) representation of a $\text{MA}(\infty)$ process. Let's consider the following stationary AR (1) process:

$$x_t = \phi_o + \phi_1 x_{t-1} + a_t$$
$$x_t - \mu = \phi_o - \mu + \phi_1 \mu + \phi_1 x_{t-1} - \phi_1 \mu + a_t$$
$$(1 - \phi_1 L)(x_t - \mu) = \phi_o - \mu + \phi_1 \mu + a_t$$

Now, by subtracting the long-run mean from the response variable (x_t), the process now has zero long-run (unconditional/marginal) mean.

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$$\phi_o - \mu + \phi_1\mu = 0$$

$$\Rightarrow \mu = \frac{\phi_o}{1-\phi_1}$$

$$\Rightarrow \phi_1 \neq 1$$

Next, the process can be further simplified as follows:

$$(1-\phi_1 L)(x_t - \mu) = (1-\phi_1 L)z_t = a_t$$

$$z_t = \frac{a_t}{1-\phi_1 L}$$

For a stationary process, the $\|\phi_1\| < 1$

$$z_t = \frac{a_t}{1-\phi_1 L} = (1 + \phi_1 L + \phi_1^2 L^2 + \phi_1^3 L^3 + ... + \phi_1^N L^N + ...)a_t$$

In sum, using the AR(1) model, we are able to represent this $\mathrm{MA}(\infty)$ model using a smaller storage requirement.

We can generalize the procedure for a stationary AR(p) model, and assuming an $\mathrm{MA}(\infty)$ representation exists, the MA coefficients' values are solely determined by the AR coefficient values:

$$x_t = \phi_o + \phi_1 x_{t-1} + \phi_2 x_{t-2} + ... + \phi_p x_{t-p} + a_t$$

$$x_t - \mu = \phi_o - \mu + \phi_1\mu + \phi_2\mu + ... + \phi_p\mu + \phi_1 x_{t-1} - \phi_1\mu + \phi_2 x_{t-2} - \phi_2\mu + ... + \phi_p x_{t-p} - \phi_p\mu + a_t$$

$$(1 - \phi_1 L - \phi_2 L^2 - ... - \phi_p L^p)(x_t - \mu) = \phi_o - \mu + \phi_1\mu + \phi_2\mu + .. + \phi_p\mu + a_t = a_t$$

Once again, by design, the long-run mean of the revised model is zero.

$$\phi_o - \mu + \phi_1\mu + \phi_2\mu + .. + \phi_p\mu = 0$$

$$\Rightarrow \mu = \frac{\phi_o}{1 - \phi_1 - \phi_2 - ... - \phi_p}$$

$$\Rightarrow \sum_{i=1}^{p} \phi_i \neq 1$$

Hence, the process can be represented as follows:

$$(1 - \phi_1 L - \phi_2 L^2 - ... - \phi_p L^p)z_t = a_t$$

$$(x_t - \mu) = z_t = \frac{a_t}{1 - \phi_1 L - \phi_2 L^2 - ... - \phi_p L^p} = \frac{a_t}{(1 - \lambda_1 L)(1 - \lambda_2 L)..(1 - \lambda_p L)}$$

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By having $\|\lambda_i\| < 1, \ \forall i \in \{1, 2, .., p\}$, we can use the partial-fraction decomposition and the geometric series representation; we then construct the algebraic equivalent of the $\mathrm{MA}(\infty)$ representation.

Hint: By now, this formulation looks enough like what we have done earlier in the MA technical note, since we inverted a finite order MA process into an equivalent representation of $AR(\infty)$.

The key point is being able to convert a stationary, finite-order AR process into an algebraically equivalent $\mathrm{MA}(\infty)$ representation. This property is referred to as **causality**.

Causality

Definition: A linear process $\{X_t\}$ is causal (strictly, a causal function of $\{a_t\}$) if there is an equivalent $\mathrm{MA}(\infty)$ representation.

$$X_t = \Psi(L)a_t = \sum_{i=0}^{\infty} \psi_i L^i a_t$$

Where:

$$\sum_{i=1}^{\infty} \|\psi_i\| < \infty$$

Causality is a property of both $\{X_t\}$ and $\{a_t\}$.

In plain words, the value of $\{X_t\}$ is solely dependent on the past values of $\{a_t\}$.

IMPORTANT: An AR(p) process is **causal** (with respect to $\{a_t\}$) if and only if the characteristics roots (i.e. $\frac{1}{\lambda_i}$) fall outside the unit circle (i.e. $\frac{1}{\|\lambda_i\|} > 1 \Rightarrow \|\lambda_i\| < 1$).

Let's consider the following example:

$$(1 - \phi L)(x_t - \mu) = (1 - \phi L)z_t = a_t$$
$$\|\phi\| > 1$$
$$z_t = \phi z_{t-1} + a_t$$

Now, let's re-organize the terms in this model:

$$z_{t-1} = \frac{1}{\phi}(z_t - a_t)$$
$$z_{t-1} = \psi z_t + a_t^{"} \qquad\qquad \|\psi\| < 1$$

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$$z_t = \psi(\psi\, z_{t+2} + a''_{t+2}) + a''_{t+1} = \psi^2 z_{t+2} + \psi a''_{t+2} + a''_{t+1}$$

$$z_t = \psi^3 z_{t+3} + \psi^2 a''_{t+3} + \psi a''_{t+2} + a''_{t+1}$$

$$z_t = \psi^N z_{t+N} + \psi^{N-1} a''_{t+N} + \psi^{N-2} a''_{t+N-1} + \ldots + \psi a''_{t+2} + a''_{t+1}$$

$$z_t = a''_{t+1} + \psi a''_{t+2} + \psi^2 a''_{t+3} + \ldots + \psi^N a''_{t+N+1} + \ldots$$

The process above is non-causal, as its values depend on future values of $\{a''_t\}$ observations. However, it is also stationary.

Going forward, for an AR (and ARMA) process, stationarity is not sufficient by itself; the process must be causal as well. For all our future discussions and application, we shall only consider stationary causal processes.

Stability

Similar to what we did in the moving average model paper, we will now examine the long-run marginal (unconditional) mean and variance.

(1) Let's assume the long-run mean (μ) exists, and:

$$E[x_t] = E[x_{t-1}] = \ldots = E[x_{t-p}] = \mu$$

Now, subtract the long-run mean from all output variables:

$$x_t - \mu + \mu = \phi_o + \phi_1(x_{t-1} - \mu) + \phi_1\mu + \phi_2(x_{t-2} - \mu) + \phi_2\mu + \ldots + \phi_p(x_{t-p} - \mu) + \phi_p\mu + a_t$$

$$(x_t - \mu) = \phi_1(x_{t-1} - \mu) + \phi_2(x_{t-2} - \mu) + \ldots + \phi_p(x_{t-p} - \mu) + a_t$$
$$+ \phi_o - \mu(1 - \phi_1 - \phi_2 - \ldots - \phi_p)$$

Take the expectation from both sides:

$$E[x_t - \mu] = E[\phi_1(x_{t-1} - \mu) + \phi_2(x_{t-2} - \mu) + \ldots + \phi_p(x_{t-p} - \mu) + a_t]$$
$$+ \phi_o - \mu(1 - \phi_1 - \phi_2 - \ldots - \phi_p)$$

$$0 = \phi_o - \mu(1 - \phi_1 - \phi_2 - \ldots - \phi_p)$$

$$\mu = \frac{\phi_o}{1 - \phi_1 - \phi_2 - \ldots - \phi_p}$$

$$\sum_{i=1}^{p} \phi_i \neq 1$$

In sum, for the long-run mean to exist, the sum of values of the AR coefficients can't be equal to one.

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(2) To examine the long-run variance of an AR process, we'll use the equivalent $\mathrm{MA}(\infty)$ representation and examine its long-run variance.

$$x_t - \mu = y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \phi_3 y_{t-3} + \ldots + \phi_p y_{t-p} + a_t$$

$$(1 - \phi_1 L - \phi_2 L^2 - \ldots - \phi_p L^p) y_t = a_t$$

$$y_t = \frac{a_t}{1 - \phi_1 L - \phi_2 L^2 - \ldots - \phi_p L^p}$$

Using partial-fraction decomposition:

$$y_t = \left[\frac{c_1}{1 - \lambda_1 L} + \frac{c_2}{1 - \lambda_2 L} + \ldots + \frac{c_p}{1 - \lambda_p L} \right] a_t$$

For a stable MA process, all characteristics roots (i.e. $\frac{1}{\lambda_i}$) must fall outside the unit circle (i.e.

$\|\lambda_i\| < 1$):

$$y_t = \left[(c_1 + c_2 + \ldots + c_p) + (c_1 \lambda_1 + c_2 \lambda_2 + \ldots + c_p \lambda_p) L + (c_1 \lambda_1^2 + c_2 \lambda_2^2 + \ldots + c_p \lambda_p^2) L^2 + \ldots \right] a_t$$

Next, let's examine the convergence property of the MA representation:

$$\lim_{k \to \infty} c_1 \lambda_1^k + c_2 \lambda_2^k + \ldots + c_p \lambda_p^k = 0$$

Finally, the long-run variance of an infinite MA process exists if the sum of its squared coefficients is finite.

$$\mathrm{Var}[x_{T+k}]_{k \to \infty} = (1 + (c_1 + c_2 + \ldots + c_p)^2 + (c_1 \lambda_1 + c_2 \lambda_2 + \ldots + c_p \lambda_p)^2 + \ldots$$

$$+ (c_1 \lambda_1^k + c_2 \lambda_2^k + \ldots + c_p \lambda_p^k)^2 + \ldots) \sigma^2$$

$$\sum_{i=1}^{\infty} (c_1 \lambda_1^i + c_2 \lambda_2^i + \ldots + c_p \lambda_p^i)^2 = \sum_{i=1}^{\infty} (\sum_{j=1}^{p} c_j \lambda_j^i)^2 < \infty$$

Furthermore, for the AR(p) process to be causal, the sum of absolute coefficient values is finite as well.

$$\sum_{k=1}^{\infty} \|\psi_k\| = \sum_{i=1}^{\infty} \left\| \sum_{j=1}^{p} c_j \lambda_j^i \right\| < \infty$$

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Example: AR(1)

$$(1 - \phi L) y_t = a_t$$

$$y_t = \frac{a_t}{1 - \phi L} = (1 + \phi L + \phi^2 L^2 + \ldots) a_t$$

$$\mathrm{Var}[y_t] = (1 + \phi^2 + \phi^4 + \phi^6 + \ldots)\sigma^2 = \frac{\sigma^2}{1 - \phi^2}$$

Assuming all characteristic roots ($\frac{1}{\lambda_i}$) fall outside the unit circle, the AR(p) process can be viewed as a weighted sum of p-stable MA processes, so a finite long-run variance must exit.

Impulse Response Function (IRF)

Earlier, we used AR(p) characteristics roots and partial-fraction decomposition to derive the equivalent of an infinite order moving average representation. Alternatively, we can compute the impulse response function (IRF) and find the MA coefficients' values.

The impulse response function describes the model output triggered by a single shock at time t.

$$a_t = \begin{cases} 0 & t \neq 1 \\ 1 & t = 1 \end{cases}$$

$$y_1 = a_1 = 1$$
$$y_2 = \phi_1 y_1 = \phi_1$$
$$y_3 = \phi_1 y_2 + \phi_2 y_1 =$$
$$y_4 = \phi_1 y_3 + \phi_2 y_2 + \phi_3 y_1$$
$$y_5 = \phi_1 y_4 + \phi_2 y_3 + \phi_3 y_2 + \phi_4 y_1$$
$$\ldots$$
$$y_{p+1} = \phi_1 y_p + \phi_2 y_{p-1} + \phi_3 y_{p-2} + \ldots + \phi_p y_1$$
$$y_{p+2} = \phi_1 y_{p+1} + \phi_2 y_p + \phi_3 y_{p-1} + \ldots + \phi_p y_2$$
$$\ldots$$
$$y_{p+k} = \phi_1 y_{p+k-1} + \phi_2 y_{p+k-2} + \phi_3 y_{p+k-3} + \ldots + \phi_p y_k$$

The procedure above is relatively simple (computationally) to perform, and can be carried on for any arbitrary order (i.e. k).

Note: Recall the partial fraction decomposition we did earlier:

$$y_t = \left[\frac{c_1}{1 - \lambda_1 L} + \frac{c_2}{1 - \lambda_2 L} + \ldots + \frac{c_p}{1 - \lambda_p L} \right] a_t$$

We derived the values for the MA coefficients as follows:

$$y_t = \left[(c_1 + c_2 + ... + c_p) + (c_1\lambda_1 + c_2\lambda_2 + ... + c_p\lambda_p)L + (c_1\lambda_1^2 + c_2\lambda_2^2 + ... + c_p\lambda_p^2)L^2 + ...\right]a_t$$

In principle, the IRF values must match the MA coefficients values. So we can conclude:

(1) The sum of denominators (i.e. c_i) of the partial-fractions equals to one (i.e. $\sum_{i=1}^{p} c_i = y_1 = 1$).

(2) The weighted sum of the characteristics roots equals to ϕ_1 (i.e. $\sum_{i=1}^{p} c_i\lambda_i = y_2 = \phi_1$).

(3) The weighted sum of the squared characteristics roots equals to $\phi_1^2 + \phi_2$ (i.e.

$$\sum_{i=1}^{p} c_i\lambda_i^2 = y_3 = \phi_1^2 + \phi_2).$$

Forecasting

Given an input data sample $\{x_1, x_2, ..., x_T\}$, we can calculate values of the moving average process for future (i.e. out-of-sample) values as follows:

$$y_T = \phi_1 y_{T-1} + \phi_2 y_{T-2} + ... + \phi_p y_{T-p} + a_T$$

$$E[y_{T+1}] = \phi_1 y_T + \phi_2 y_{T-1} + ... + \phi_p y_{T+1-p}$$
$$E[y_{T+2}] = \phi_1 E[y_{T+1}] + \phi_2 y_T + ... + \phi_p y_{T+2-p}$$
$$= (\phi_1^2 + \phi_2)y_T + (\phi_1\phi_2 + \phi_3)y_{T-1} + ... + (\phi_1\phi_{p-1} + \phi_p)y_{T+2-p} + \phi_p y_{T+2-p}$$

We can carry this calculation to any number of steps we wish.

Next, for the forecast error:

$$\text{Var}[y_{T+1}] = \text{Var}[\phi_1 y_T + \phi_2 y_{T-1} + ... + \phi_p y_{T+1-p} + a_{T+1}] = \sigma^2$$
$$\text{Var}[y_{T+2}] = \text{Var}[\phi_1 y_{T+1} + \phi_2 y_T + ... + \phi_p y_{T+2-p} + a_{T+2}] = (1 + \phi_1^2)\sigma^2$$
$$\text{Var}[y_{T+3}] = \text{Var}[\phi_1 y_{T+2} + \phi_2 y_{T+1} + ... + \phi_p y_{T+3-p} + a_{T+3}]$$
$$= \text{Var}[\phi_1(\phi_1 y_{T+1} + \phi_2 y_T + ... + \phi_p y_{T+2-p} + a_{T+2}) + \phi_2 y_{T+1} + ... + \phi_p y_{T+3-p} + a_{T+3}]$$
$$= \text{Var}[(\phi_1^2 + \phi_2)y_{T+1} + ... + \phi_1 a_{T+2} + ... + a_{T+3}] = (1 + \phi_1^2 + (\phi_1^2 + \phi_2)^2)\sigma^2$$

As the number of steps increase, the formulas become more cumbersome. Alternately, we can use the $MA(\infty)$ equivalent representation and compute the forecast error.

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$$\text{IRF} = \{z_t\} \quad z_t = (1 + \theta_1 L + \theta_2 L^2 + ...)a_t$$

And the forecast error is expressed as follows:

$$\text{Var}[y_{T+1}] = \sigma^2$$
$$\text{Var}[y_{T+2}] = (1 + \theta_1^2)\sigma^2$$
$$\text{Var}[y_{T+3}] = (1 + \theta_1^2 + \theta_2^2)\sigma^2$$
$$....$$
$$\text{Var}[y_{T+k}] = (1 + \theta_1^2 + \theta_2^2 + ... + \theta_{k-1}^2)\sigma^2$$
$$....$$
$$\text{Var}[y_{T+k}]_{k \to \infty} = (1 + \theta_1^2 + \theta_2^2 + ...)\sigma^2$$

Note: The conditional variance grows cumulatively over an infinite number of steps to reach its long-run (unconditional) variance.

Correlogram

What do the auto-regressive (AR) correlogram plots look like? How can we identify an AR process (and its order) using only ACF or PACF plots?

First, let's examine the ACF for an AR process:

$$\text{ACF(k)} = \rho_k = \frac{\gamma_k}{\gamma_o}$$

Where:

$$\gamma_j = E[(x_t - \mu)(x_{t-j} - \mu)] \qquad \text{(covariance for lag j)}$$
$$\gamma_o = E[(x_t - \mu)^2] \qquad \text{(long-run variance)}$$

Let's first compute the auto-covariance function γ_j.

$$\gamma_1 = E[(x_t - \mu)(x_{t-1} - \mu)] = E[z_t \times z_{t-1}]$$
$$\gamma_1 = E[(\phi_1 z_{t-1} + \phi_2 z_{t-2} + .. + \phi_p z_{t-p} + a_t) \times z_{t-1}] = \phi_1 \gamma_o + \phi_2 \gamma_1 + \phi_3 \gamma_2 + ... + \phi_p \gamma_{p-1}$$
$$(1 - \phi_2)\gamma_1 - \phi_3 \gamma_2 - ... - \phi_p \gamma_{p-1} = \phi_1 \gamma_o$$

$$\gamma_2 = E[(\phi_1 z_{t-1} + \phi_2 z_{t-2} + .. + \phi_p z_{t-p} + a_t) \times z_{t-2}] = (\phi_1 + \phi_3)\gamma_1 + \phi_2 \gamma_o + \phi_4 \gamma_2 + ... + \phi_p \gamma_{p-2}$$
$$-(\phi_1 + \phi_3)\gamma_1 - (\phi_4 - 1)\gamma_2 - \phi_5 \gamma_3 - ... - \phi_p \gamma_{p-2} = \phi_2 \gamma_o$$

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Next, for the 3rd lag covariance;

$$\gamma_3 = E[(\phi_1 z_{t-1} + \phi_2 z_{t-2} + .. + \phi_p z_{t-p} + a_t) \times z_{t-3}]$$

$$\gamma_3 = \phi_1 \gamma_2 + \phi_2 \gamma_1 + \phi_3 \gamma_o + \phi_4 \gamma_1 + \phi_5 \gamma_2 + \phi_6 \gamma_3 + ... + \phi_p \gamma_{p-2}$$

$$-(\phi_2 + \phi_4)\gamma_1 - (\phi_1 + \phi_5)\gamma_2 - (\phi_6 - 1)\gamma_3 - \phi_7 \gamma_4 - ... - \phi_p \gamma_{p-2} = \phi_3 \gamma_o$$

In sum, for an AR(p) process, we need to construct and solve p-1 linear systems to compute the values of the first p-1 auto-covariances.

$$
\begin{bmatrix}
1-\varphi_2 & -\varphi_3 & -\varphi_4 & -\varphi_5 & -\varphi_6 & . & -\varphi_{p-1} & -\varphi_p \\
-(\varphi_1+\varphi_3) & 1-\varphi_4 & -\varphi_5 & -\varphi_6 & -\varphi_7 & . & -\varphi_p & 0 \\
-(\varphi_2+\varphi_4) & -(\varphi_1+\varphi_5) & 1-\varphi_6 & -\varphi_7 & -\varphi_8 & . & 0 & 0 \\
-(\varphi_3+\varphi_5) & -(\varphi_2+\varphi_6) & -(\varphi_1+\varphi_7) & 1-\varphi_8 & -\varphi_9 & . & 0 & 0 \\
-(\varphi_4+\varphi_6) & -(\varphi_3+\varphi_7) & -(\varphi_2+\varphi_8) & -(\varphi_1+\varphi_9) & 1-\varphi_{10} & . & 0 & 0 \\
. & . & . & . & . & . & . & . \\
-(\varphi_{p-3}+\varphi_{p-1}) & -(\varphi_{p-4}+\varphi_p) & -\varphi_{p-5} & -\varphi_{p-4} & -\varphi_{p-5} & . & 0 & 0 \\
-(\varphi_{p-2}+\varphi_p) & -\varphi_{p-3} & -\varphi_{p-4} & -\varphi_{p-5} & -\varphi_{p-6} & . & -\varphi_1 & 1
\end{bmatrix}
\begin{bmatrix}
\gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \gamma_5 \\ . \\ \gamma_{p-2} \\ \gamma_{p-1}
\end{bmatrix}
=
\begin{bmatrix}
\varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ \varphi_5 \\ . \\ \varphi_{p-2} \\ \varphi_{p-1}
\end{bmatrix} \times \gamma_o
$$

The auto-covariance for lags greater than p-1 is computed iteratively as follows:

$$\gamma_p = \phi_1 \gamma_{p-1} + \phi_2 \gamma_{p-2} + ... + \phi_{p-1} \gamma_1 + \phi_p \gamma_o$$

$$\gamma_{p+1} = \phi_1 \gamma_p + \phi_2 \gamma_{p-1} + ... + \phi_{p-1} \gamma_2 + \phi_p \gamma_1$$

$$\gamma_{p+2} = \phi_1 \gamma_{p+1} + \phi_2 \gamma_p + ... + \phi_{p-1} \gamma_3 + \phi_p \gamma_2$$

...

$$\gamma_{p+k} = \phi_1 \gamma_{p+k-1} + \phi_2 \gamma_{p+k-2} + ... + \phi_{p-1} \gamma_{k+1} + \phi_p \gamma_k$$

Example: For an AR(5) process, the linear system of equations of the auto-covariance functions is expressed below:

$$
\begin{bmatrix}
1-\varphi_2 & -\varphi_3 & -\varphi_4 & -\varphi_5 \\
-(\varphi_1+\varphi_3) & 1-\varphi_4 & -\varphi_5 & 0 \\
-(\varphi_2+\varphi_4) & -(\varphi_1+\varphi_5) & 1-\varphi_6 & 0 \\
-(\varphi_3+\varphi_5) & -\varphi_2 & -\varphi_1 & 1
\end{bmatrix}
\begin{bmatrix}
\gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4
\end{bmatrix}
=
\begin{bmatrix}
\varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4
\end{bmatrix} \times \gamma_o
$$

Q: what do they look like in the ACF plot?

- Due to the causality effect, ACF values of a true AR process don't drop to zero at any lag number, but rather tail exponentially.
- This property helps us to qualitatively identify the AR/ARMA (vs. MA) process in the ACF plot.
- Determining the actual order (i.e. p) of the underlying AR process is, in most cases, difficult.

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Example: Let's consider the AR(1) process:

$$z_t = \varphi z_{t-1} + a_t$$

$$\gamma_1 = E[z_t \times z_{t-1}] = \varphi \gamma_o$$
$$\gamma_2 = E[z_t \times z_{t-2}] = \varphi \gamma_1 = \varphi^2 \gamma_o$$
$$\gamma_3 = \varphi \gamma_2 = \varphi^3 \gamma_o$$
$$\dots$$
$$\gamma_k = \varphi \gamma_{k-1} = \varphi^k \gamma_o$$

The ACF for an AR(1) process can be expressed as follows:

$$\text{ACF(k)} = \frac{\gamma_k}{\gamma_o} = \varphi^k$$

The ACF values don't drop to zero at any lag number, but rather decline exponentially.

Q: What about a higher order AR process?

The ACF plot can get increasingly more complex, but it will always tail exponentially. This is due to the model's causal property. We can tell the difference between an MA process and an AR/ARMA process by this qualitative difference.

We need a different tool or plot to help identify the exact order of the AR process and its order: a plot that drops to zero after the p-th lags when the true model is AR(p). This tool or plot is the partial auto-correlation plot (PACF).

Partial auto-correlation function (PACF)

The partial auto-correlation function (PACF) is interpreted as the correlation between x_t and x_{t-h}, where the linear dependency of the intervening lags ($x_{t-1}, x_{t-2}, \dots, x_{t-h+1}$) has been removed.

$$\text{PACF}(h) = Corr(x_t, x_{t-h} \mid x_{t-1}, x_{t-2}, \dots, x_{t-h+1})$$

Note that this is also how the parameters of a multiple linear regression (MLR) models are interpreted.

Example:

$$x_t = \beta_o + \beta_1 t^2$$
$$x_t = \beta_o + \beta_1 t + \beta_2 t^2$$

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In the first model, β_1 is interpreted as the linear dependency between t^2 and x_t. In the second model, the β_2 is interpreted as the linear dependency between t^2 and x_t, but with the dependency between t and x_t already accounted for.

In sum, the PACF has a very similar interpretation as the coefficients in the multiple regression situations and the PACF values are estimated using those coefficient values.

(1) Construct a series of regression models and estimate the parameters values:

$$x_t = \beta_{0,1} + \beta_{1,1} x_{t-1} + a_t$$
$$x_t = \beta_{0,2} + \beta_{1,2} x_{t-1} + \beta_{2,2} x_{t-2} + a_t$$
$$x_t = \beta_{0,3} + \beta_{1,3} x_{t-1} + \beta_{2,3} x_{t-2} + \beta_{3,3} x_{t-3} + a_t$$
$$x_t = \beta_{0,4} + \beta_{1,4} x_{t-1} + \beta_{2,4} x_{t-2} + \beta_{3,4} x_{t-3} + \beta_{4,4} x_{t-4} + a_t$$
$$...$$
$$x_t = \beta_{0,k} + \beta_{1,k} x_{t-1} + \beta_{2,k} x_{t-2} + \beta_{3,k} x_{t-3} + ... + \beta_{k,k} x_{t-k} + a_t$$

(2) The PACF(k) is estimated by $\beta_{k,k}$.

Notes:

(1) To estimate the PACF of the first k-lags, we'd need to solve k regression models, which can be slow for larger data sets. A number of algorithms (e.g. Durbin-Levenson algorithm and Yule-Walker estimations) can be employed to expedite the calculations.

(2) The PACF can be calculated from the sample auto-covariance. For example, to estimate the PACF(2), we solve the following system:

$$\begin{bmatrix} \gamma_o & \gamma_{-1} \\ \gamma_1 & \gamma_o \end{bmatrix} \begin{bmatrix} \beta_{1,2} \\ \beta_{2,2} \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}$$

For PACF (3), we solve the following system:

$$\begin{bmatrix} \gamma_o & \gamma_{-1} & \gamma_{-2} \\ \gamma_1 & \gamma_o & \gamma_{-1} \\ \gamma_2 & \gamma_1 & \gamma_o \end{bmatrix} \begin{bmatrix} \beta_{1,3} \\ \beta_{2,3} \\ \beta_{3,3} \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix}$$

Using the Durbin-Levenson algorithm improves the calculation speed dramatically by re-using prior calculations to estimate current ones.

$$\gamma_j = E[(x_t - \mu)(x_{t-j} - \mu)]$$

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By definition, the auto-covariance of lag order zero (γ_o) is the unconditional (marginal) variance.

By design, for a true AR(p) process, the corresponding PACF plot drops to zero after p-lags. On the other hand, the ACF plot tails (declines) exponentially.

Using only the PACF plot, I should be able to construct an AR model for any process, right? **No.**

The PACF plot mainly examines whether the underlying process is a true AR process and identifies the order of the model.

Conclusion

To recap, in this paper, we laid the foundation for a slightly more complex model: the autoregressive model (AR). First, we presented the AR process as a restricted form of an infinite-order MA process. Next, armed with a few mathematical tricks (i.e. IRF, partial fraction decomposition and geometric series), we tackled many more complex characteristics of this process (e.g. forecasting, long-run variance, etc.) by representing it as an MA process.

Later on, we introduced a new concept: Causality. A process is defined as causal if and only if its values $\{X_t\}$ are dependent on the process's past shocks/innovations $\{a_t, a_{t-1}, a_{t-2}, ...\}$. We showed that stationarity is not a sufficient condition for our models; they must be causal as well.

Finally, we delved into AR process identification using correlogram (i.e. ACF and PACF) plots. We showed that the ACF of an AR process does not drop to zero, but rather tails exponentially in all cases. Furthermore, we looked into PACF plots and outlined that fact that PACF, by design, drops to zero after p-lags for a true AR process.

As we go on to discuss more advanced models in future technical notes, we will often refer to the MA and AR processes and the material presented here.

References

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